Solution Sheet 2

Exercise 1. (for credit, due on 28 September) Consider the rational function $f(z) = \frac{(z+1)^2}{z^4}$.

1. Show that f extends to a holomorphic map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$.
2. Compute the ramification points (including multiplicities) and branch values of f.

Solution 1.

1. On $\mathbb{C}\backslash\{0\}$, f is holomorphic. At $z = 0$, f has a pole of order 4. We can extend f holomorphically across 0 by setting $f(0) = \infty$. Indeed, then
$$\frac{1}{f(z)} = \frac{z^4}{(z+1)^2}$$
is holomorphic at $z = 0$. At $z = \infty$, we use the local coordinate $w = 1/z$. Then
$$f(1/w) = (1+w)^2 w^2, \tag{1}$$
which is holomorphic at $w = 0$. We can therefore extend f holomorphically to $\mathbb{CP}^1$ by setting $f(\infty) = 0$.
2. Recall that p is a ramification point if in a local coordinate z at p we can write $f(z) = f(p) + cz^{m_p} + \cdots$ with $m_p \geq 2$. If $p \in \mathbb{C}$ and $f(p) \in \mathbb{C}$, then this is equivalent to $f'(p) = 0$. We have
$$f'(z) = -\frac{2(z+1)(z+2)}{z^5}.$$
Hence $z = -1$ and $z = -2$ are ramification points of f with multiplicity 2, and branch values $f(-1) = 0$ and $f(-2) = 1/16$. For poles and ∞, we use suitable coordinates at the domain and target. At $z = \infty$, we use the coordinate $w = 1/z$ and observe from (1) that $f(1/w)$ has a zero of order 2 at $w = 0$. Hence $e_\infty = 2$, with branch value $f(\infty) = 0$. At $z = 0$, f has a pole of order 4. Hence $m_0 = 4$ with branch value $f(0) = \infty$.

Exercise 2. Let $f : X \to Y$ be a non-constant holomorphic map between connected Riemann surfaces.

1. Show that the set of ramification points of f is discrete in X.
2. Assume that f is also proper (i.e. the inverse image of a compact set is compact). Show that the set of branching points is discrete.
3. Assume that $X = Y$ and $f \neq \mathrm{id}_X$. Show that the set of fixed points $\{p \in X : f(p) = p\}$ is discrete.
4. Give an example where the set of ramification points is infinite and the set of branching points is not discrete. **Hint:** Trigonometric functions.

Solution 2.

1. Fix $p \in X$. Choose connected coordinate charts (U, z) about p and (V, w) about $f(p)$ with $f(U) \subset V$. Set $g := w \circ f \circ z^{-1}$. Then g is holomorphic, and p is a ramification point of f iff $g'(z(p)) = 0$. The function g' is not identically zero on $D := z(U)$. Indeed, if $g' \equiv 0$ on D, then g is constant on D, hence f is constant on the nonempty open set U. By the identity theorem on the connected surface X, this forces f to be constant on all of X, a contradiction. Thus $g' \not\equiv 0$, and the zeros of the nontrivial holomorphic function g' are isolated in D. Pulling back by z, each ramification point of f is isolated in X.
2. Let $q \in Y$ be a branching point. Choose $K \ni q$ a compact neighborhood of q in Y. We know that the set of ramification points in $f^{-1}(K)$ is discrete, hence finite by compactness of $f^{-1}(K)$. Consequently, there are finitely many branching points in K, so q is isolated. This shows that the set of branching points is discrete.

(3) In a local coordinate z on X, define $g = z \circ f \circ z^{-1} - \mathrm{id}$. Its zero set is the set of fixed points of f is that chart. Since f is not the identity, we have $g \not\equiv 0$. Again by the identity theorem, the zeros of g cannot accumulate, hence are isolated. We then cover X by charts.

(4) Consider the holomorphic function on the complex plane given by

$$f(z) = e^z(1 + \sin(z)).$$

Its derivative is

$$f'(z) = e^z(1 + \sin(z) + \cos(z)).$$

The ramification points are exactly the solutions of $\sin(z) + \cos(z) = -1$. We find that the set of ramification points is

$$\{\pi + 2\pi\mathbb{Z}\} \cup \{-\frac{\pi}{2} + 2\pi\mathbb{Z}\},$$

which is infinite and discrete. Now we note that $f(-\frac{\pi}{2} + 2\pi n) = 0$ and $f(\pi + 2\pi n) = e^{\pi}e^{2\pi n}$ for all $n \in \mathbb{Z}$. Because $e^{\pi}e^{2\pi n} \to 0$ as $n \to -\infty$, infinitely many branching points accumulate at 0. Since 0 is also a branching point, we conclude that the set of branching points is not discrete.

Exercise 3. Show that a bijective holomorphic map between two Riemann surfaces is a biholomorphism.

Solution 3. Let $f : X \to Y$ be a bijective holomorphic map of Riemann surfaces. Fix $p \in X$ and choose local charts z at p with $z(p) = 0$ and w at $f(p)$ with $w(f(p)) = 0$. In these coordinates $g := w \circ f \circ z^{-1}$ has a Taylor expansion $g(\xi) = a_k\xi^k + a_{k+1}\xi^{k+1} + \cdots$ for some $k \geq 1$, $a_k \neq 0$. If $k \geq 2$, then near 0 the map $\xi \mapsto g(\xi)$ is k-to-1, contradicting injectivity. Thus $k = 1$ and so $g'(0) \neq 0$ and $f'(p) \neq 0$. Since p was arbitrary, $f'(p) \neq 0$ for all $p \in X$. The analytic inverse function theorem then implies that for each p there are neighborhoods $U_p \ni p$ and $V_p \ni f(p)$ such that $f|_{U_p}$ is a biholomorphism. On overlaps $V_p \cap V_q$, the local inverses agree because injectivity of f gives a unique preimage. Therefore the local inverses glue to a global holomorphic inverse.

Exercise 4 (Schwarz lemma)**.** Let $\mathbb{D} = \{z \in \mathbb{C} \;:\; |z| < 1\}$ be the unit disk and suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$. Show that $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \leq 1$. Show that if moreover $|f(z_0)| = |z_0|$ for some non-zero $z_0 \in \mathbb{D}$ or $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.

Solution 4. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. The function

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases}$$

is holomorphic on $\mathbb{D}$. Fix r with $0 < r < 1$. On $|z| = r$ we have $|f(z)| \leq 1$, hence

$$|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

We apply the maximum modulus principle to the holomorphic function g on $|z| \leq r$ to get

$$|g(z)| \leq \max_{|z|=r} |g(z)| \leq \frac{1}{r}$$

for all $|z| \leq r$. Since this holds for all $r < 1$, we obtain $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Moreover, $|f'(0)| = |g(0)| \leq 1$. This proves the first part. If $|f(z_0)| = |z_0|$ for some $0 \neq z_0 \in \mathbb{D}$, then $|g(z_0)| = 1$. If $|f'(0)| = 1$, then $|g(0)| = 1$. In either case, a holomorphic function g on $\mathbb{D}$ with $|g| \leq 1$ that attains its maximum modulus 1 at a point in $\mathbb{D}$ must be constant. Hence $g(z) = a$ for some $a \in \mathbb{C}$ with $|a| = 1$. It follows that $f(z) = az$ for all $z \in \mathbb{D}$.

Exercise 5. For a Riemann surface X, the automorphism group $\mathrm{Aut}(X)$ is the group of biholomorphisms $f : X \to X$. Denote by $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ the unit disk and by $\mathbb{H} = \{z \in \mathbb{C} \ : \ \Im(z) > 0\}$ the upper half-plane. Show that:

$$\begin{aligned}
\mathrm{Aut}(\mathbb{CP}^1) &= \left\{ z \mapsto \frac{az+b}{cz+d} \ : a,b,c,d \in \mathbb{C} \text{ with } ad-bc \neq 0 \right\};\\
\mathrm{Aut}(\mathbb{C}) &= \{ z \mapsto az+b \ : \ a,b \in \mathbb{C} \text{ with } a \neq 0 \};\\
\mathrm{Aut}(\mathbb{D}) &= \left\{ z \mapsto \frac{az+b}{\bar{b}z+\bar{a}} \ : \ a,b \in \mathbb{C} \text{ with } |a|^2-|b|^2 = 1 \right\};\\
\mathrm{Aut}(\mathbb{H}) &= \left\{ z \mapsto \frac{az+b}{cz+d} \ : \ a,b,c,d \in \mathbb{R} \text{ with } ad-bc > 0 \right\}.
\end{aligned}$$

Solution 5. Sphere. Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a biholomorphism. We proved in exercise sheet 1 (Ex. 5) that f is a rational function, that is, $f(z) = p(z)/q(z)$ where p and q are polynomials. Let $d = \max\{\deg(p), \deg(q)\}$. If $d \geq 2$, then for generic $w \in \mathbb{CP}^1$ the equation $p(z) - wq(z) = 0$ has d solutions (counting multiplicities). Hence f cannot be injective. Thus $d = 1$ and hence p and q must be linear, which gives $f(z) = (az+b)/(cz+d)$ for some $a, b, c, d \in \mathbb{C}$. If $ad - bc = 0$, then the row vectors (a, b) and (c, d) are linearly dependent, that is, $(a, b) = \lambda(c, d)$ for some $\lambda \in \mathbb{C}$. We then get

$$f(z) = \frac{az+b}{cz+d} = \frac{\lambda(cz+d)}{cz+d} = \lambda,$$

so f is constant, in particular not injective. Therefore we must have $ad - bc \neq 0$.

Plane. Let $f : \mathbb{C} \to \mathbb{C}$ be a biholomorphism. We define $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ by

$$F|_{\mathbb{C}} = f, \quad F(\infty) = \infty.$$

We claim that F is a biholomorphism of the sphere, fixing ∞. Assuming this, by the previous calculation F must have the form $F(z) = (az+b)/(cz+d)$ with $ad-bc \neq 0$. The condition $F(\infty) = \infty$ forces $c = 0$. Hence

$$F(z) = \frac{a}{d}z + \frac{b}{d} = a'z + b', \quad \text{with}\, a' \neq 0.$$

To verify the claim, we first show that $|f(z)| \to \infty$ as $|z| \to \infty$. Suppose, to the contrary, that there is a sequence $z_k \to \infty$ such that $f(z_k)$ remains bounded. By compactness (Bolzano-Weierstrass theorem), there is a subsequence along which $f(z_k)$ converges to some value $v \in \mathbb{C}$. Since f is a bijection of $\mathbb{C}$, we have $v = f(u)$ for some $u \in \mathbb{C}$. Since f^{-1} is continuous

$$z_k = f^{-1}(f(z_k)) \to f^{-1}(v) = u,$$

contradicting $z_k \to \infty$. Now we use the chart $w = 1/z$ near ∞. For $w \neq 0$ we define $g(w) = 1/f(1/w))$. By the previous step, $f(1/w) \to \infty$ as $w \to 0$. This gives $g(w) \to 0$ as $w \to 0$, so g is bounded near 0. By the removable singularity theorem, g extends holomorphically across 0 with $g(0) = 0$. Equivalently, F is holomorphic at ∞ with $F(\infty) = \infty$.

Disk. Let $f : \mathbb{D} \to \mathbb{D}$ be a biholomorphism. We take $\alpha = f^{-1}(0) \in \mathbb{D}$ and define the map

$$\phi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}.$$

The map $\phi_\alpha(z)$ is an automorphism of $\mathbb{D}$ with $\phi_\alpha(\alpha) = 0$. Indeed, it is clearly holomorphic and

$$1 - |\phi_\alpha(z)|^2 = \frac{(1-|\alpha|^2)(1-|z|^2)}{|1-\bar{\alpha}z|^2}$$

shows that $|\phi_\alpha(z)| < 1 \iff |z| < 1$ and $|\phi_\alpha(z_0)| = 1 \iff |z_0| = 1$. Hence $\phi_\alpha(\mathbb{D}) = \mathbb{D}$ and the inverse

$$\phi_\alpha^{-1}(w) = \frac{w + \alpha}{1 + \bar{\alpha} w}$$

is also holomorphic. We set $g = f \circ \phi_\alpha^{-1}$, which fixes 0. Schwarz lemma (Ex. 4) gives $|g'(0)| \leq 1$ and $|(g^{-1})'(0)| \leq 1$. Since $(g^{-1})'(0) = 1/g'(0)$, we have $|g'(0)| \geq 1$. Therefore $|g'(0)| = 1$ and Schwarz lemma yields $g(w) = \lambda w$ for some λ of modulus $|\lambda| = 1$. Consequently,

$$f(z) = g(\phi_\alpha(z)) = \lambda \frac{z - \alpha}{1 - \bar{\alpha} z}.$$

To bring this equation into the form $(az + b)/(\bar{b} z + \bar{a})$ with $|a|^2 - |b|^2 = 1$, we must rescale. For a given $|\lambda| = 1$, we write $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$. We set

$$a = \frac{e^{i\theta/2}}{\sqrt{1 - |\alpha|^2}}, \quad b = -\frac{e^{i\theta/2} \alpha}{\sqrt{1 - |\alpha|^2}}.$$

A calculation shows that $|a|^2 - |b|^2 = 1$ and

$$\lambda \frac{z - \alpha}{1 - \bar{\alpha} z} = \frac{az + b}{\bar{b} z + \bar{a}}.$$

Half-plane. Recall the Cayley transform

$$C(z) = \frac{z - i}{z + i}, \quad C^{-1}(w) = i \frac{1 + w}{1 - w}.$$

The Möbius matrices of these maps are

$$K = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \quad K^{-1} = \frac{1}{2i} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}.$$

Let $f : \mathbb{H} \to \mathbb{H}$ be a biholomorphism. We set $g = C \circ f \circ C^{-1} \in \mathrm{Aut}(\mathbb{D})$. By the previous calculation we can write

$$g(w) = \frac{az + b}{\bar{b} z + \bar{a}}, \quad a, b \in \mathbb{C} \text{ with } |a|^2 - |b|^2 = 1.$$

In Möbius matrix notation this map is

$$M = \begin{pmatrix} a & b \\ \bar{b} & \bar{a}. \end{pmatrix}$$

To find the Möbius matrix corresponding to f we need to calculate $A = K^{-1} M K$. We find

$$A = \frac{1}{2i} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a + b & i(b - a) \\ \bar{a} + \bar{b} & i(\bar{a} - \bar{b}) \end{pmatrix}.$$

Writing $a = x + iy$ and $b = y + iv$ then gives

$$A = \begin{pmatrix} x + y & u - v \\ -(u + v) & x - y \end{pmatrix}.$$

The determinant of this real matrix is $\det A = x^2 - y^2 + u^2 - v^2 = |a|^2 - |b|^2$. Therefore f is represented by a real 2×2 matrix of determinant 1. Multiplying all entries by a positive real scalar does not change the map, therefore we obtain all Möbius transformations with $ad - bc > 0$.

Exercise 6. (challenging)
What is the automorphism group of the three-punctured Riemann sphere $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$?

Solution 6. Any biholomorphism of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ extends to a Möbius map on the Riemann sphere. This extension must map the set of punctures to itself. As a Möbius map is uniquely determined by the images of three points, there are $3! = 6$ Möbius transformations that permute the three punctures. Hence the automorphism group of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ is isomorphic to the symmetric group S_3.